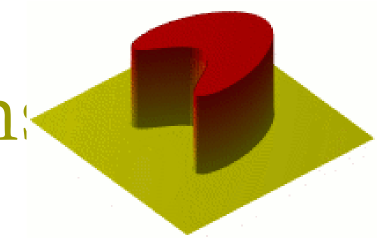
Topics on PDEs and Numerical Methods

Part 3: Finite Element Method

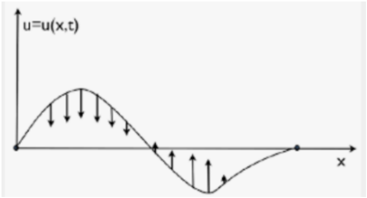
Guangming Yao
Department of Mathematics
Clarkson University

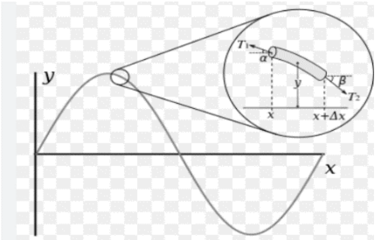
Three fundamental PDEs and solutions

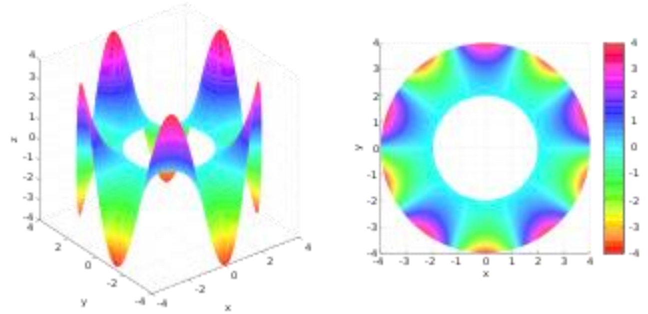


- **Heat equation (parabolic):** $u_t = u_{xx}$
  - Solution: $u = \frac{1}{2}x^2 + t$
  - Challenge: can you find another solution? $u = e^{ax+bt}$
  - Fourier, 1800's
  - Heat conduction
- **Wave equation (hyperbolic):** $u_{tt} = u_{xx}$
  - d'Alembert, 1740's, vibration of strings
- **Laplace equation (elliptic):** $u_{xx} + u_{yy} = 0$
  - Laplace, 1780's,
  - gravitation mechanical equilibrium,
  - thermal equilibrium

Vibration, standing waves in a string. The fundamental and the first 5 overtones in the harmonic series.







Laplace's equation on an annulus (inner radius $r = 2$ and outer radius $R = 4$) with Dirichlet boundary conditions $u(r{=}2) = 0$ and $u(R{=}4) = 4\sin(5\,\theta)$

Common but Challenging PDEs

- Diffusion equation

$$\nabla \cdot D\nabla C + S = 0$$

- Solid-Mechanics

$$\nabla \cdot (\rho\vec{u}\vec{u}^T) = -\nabla P + \nabla \cdot \tau + \rho g$$

- Navier-Stokes

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla \cdot (\rho\vec{u} \otimes \vec{u}) + \nabla P = \mu\nabla^2\vec{u} + \frac{\mu}{3}\nabla(\nabla \cdot \vec{u}) + \rho g$$

- Schrodinger

$$\nabla \cdot \left[-\frac{h^2}{2m^*}\nabla\psi(\vec{r})\right] + U(\vec{r})\psi(\vec{r}) = E\,\psi(\vec{r})$$

- Dynamics Electromagnetic wave equation

$$\frac{\mu\epsilon\partial^2\vec{E}(\vec{r},t)}{\partial t^2} + \frac{\mu\sigma\partial\vec{E}(\vec{r},t)}{\partial t} - \nabla^2\vec{E}(\vec{r},t) = -\frac{\mu\partial\vec{J}(\vec{r},t)}{\partial t}$$

- Boltzmann transport equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q\mathbf{E}}{\mathbf{h}} \cdot \nabla_k f = \left(\frac{\partial f}{\partial t}\right)_C$$

How to solve PDEs?

Analytically:

- Method of characteristic
- Separation of variables
- Fourier analysis----$\sin(x)$, $\cos(x)$, Bessel's function, Legendre, ..
- Eigenfunction expansion
- Problems:
  - cannot deal with complicated geometry
  - May not converge with finite terms
  - Hard to deal with nonlinear

Numerically:

Finite difference method (FDM)
Finite element method (FEM)
Finite volume method
Reduce order method

Combination of analytical and numerical methods

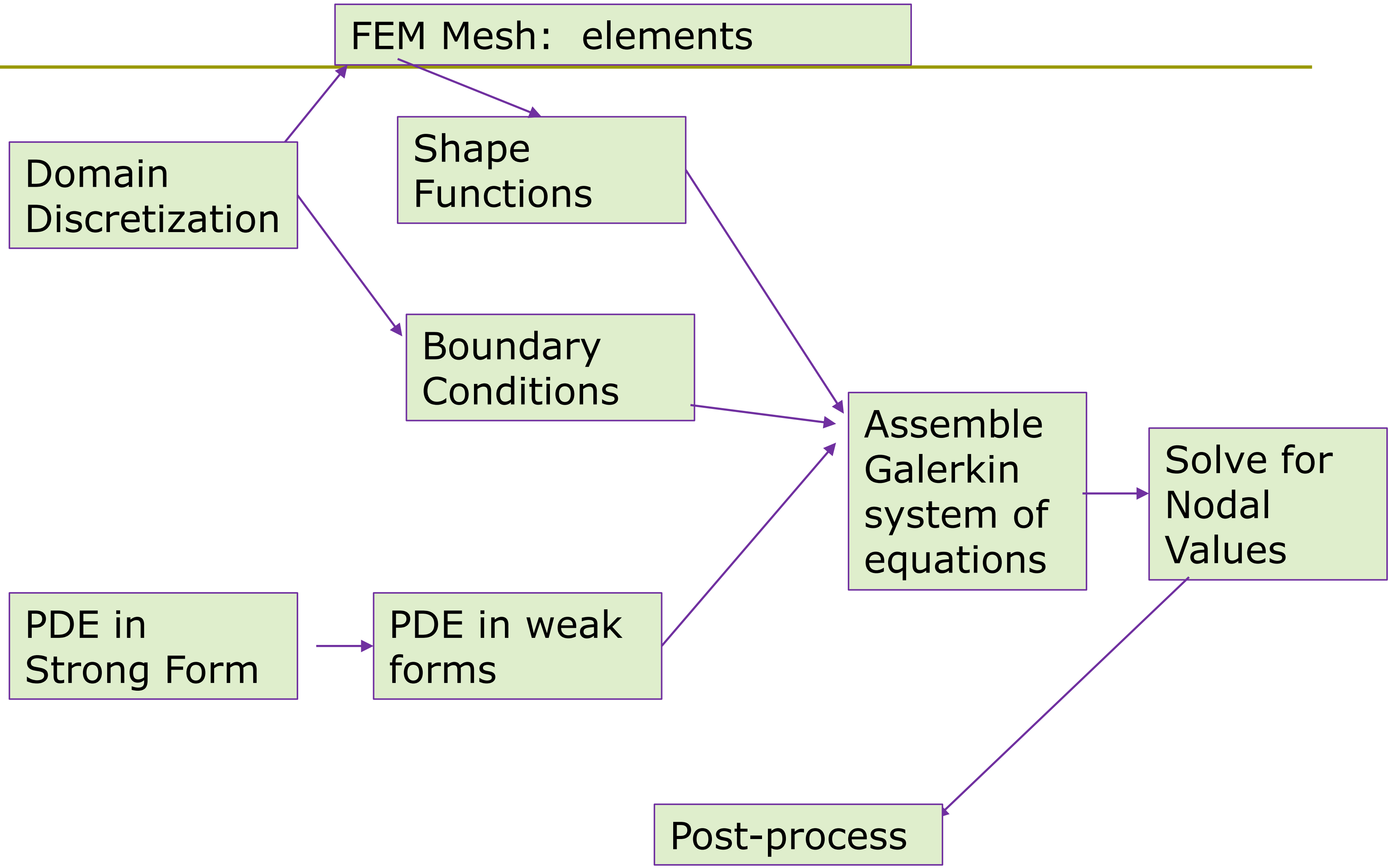
How to solve PDEs?

- Analytic solutions are possible for simple and special (idealized) cases only.
- To make use of the nature of the equations, different methods are used to solve different classes of PDEs.
- The methods discussed here are based on the finite **element** technique.
- Methods other than FEM: FDM, Spectral Method, FVM, …

- **Finite Element Method (FEM)**

- How to solve PDEs using FEM?
  - Numerical interpolation: shape functions
  - Domain discretization: mesh
  - Weak and strong forms of PDE
  - Linear or nonlinear system solver

Summary of FEM Process



Generalization of FEM

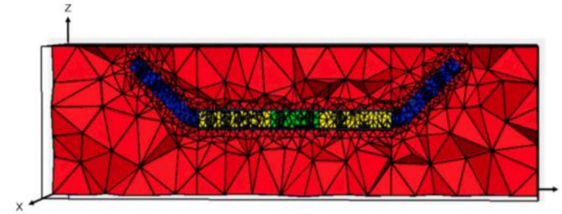
- Divide geometry into simple elements

- Finding polynomial approximation on each element: unknow $a_0, a_1, a_2, \cdots, a_n$
  - 1D: $\tilde{C}(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
  - 2D: $\tilde{C}(x) = a_0 + a_1 x + a_2 y + a_3 xy + \cdots$
  - 3D: $\tilde{C}(x) = a_0 + a_1 x + a_2 y + a_3 z + \cdots$

- Continuous across elements

- PDE → System of equations → Solve nodal values

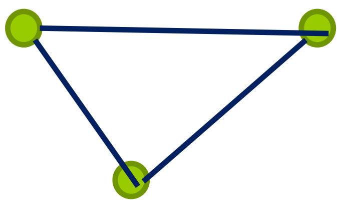
Elements

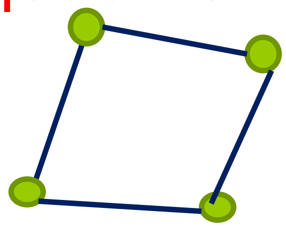


- Divide geometry into simple elements
  - Elements have nodes

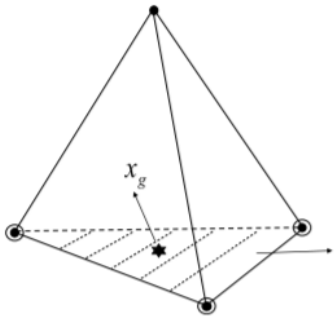
  - Elements include line segments in 1D

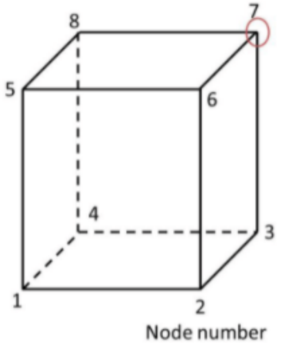
  - Elements include triangular or quadrilateral mesh in 2D





  - Elements include tetrahedrons or hexahedrons in 3D

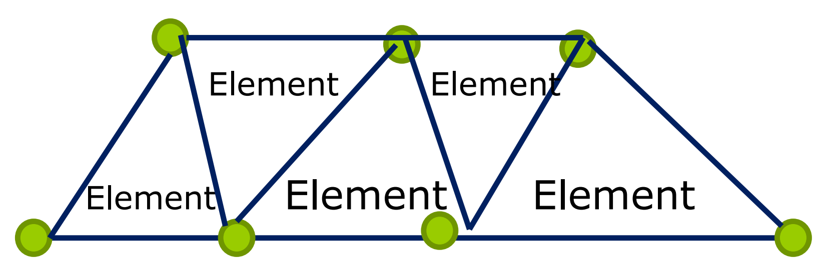




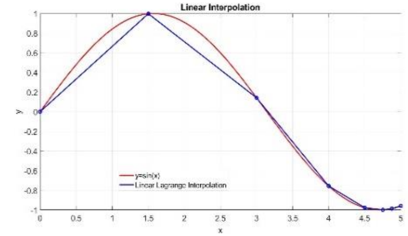
Elements

- Each element has its own coefficients:
  - One for each node: $a_0, a_1, a_2, \cdots, a_n$
- Construct shape functions, meanwhile get continuous piecewise polynomial between elements:

Write solution in terms of values at nodes → Adjacent elements share nodes → Automatically continuous between elements



Construct shape functions in 1D



- Construct shape functions by writing solutions in terms of functions' values

Consider single element

2 nodes in 1D: $\tilde{C}(x) = a_0 + a_1 x$

Find a_0 and a_1 by solving

$$C_1 = \tilde{C}(x_1) = a_0 + a_1 x_1$$
$$C_2 = \tilde{C}(x_2) = a_0 + a_1 x_2$$

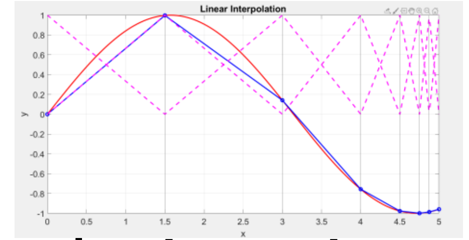
Thus, $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$

That is:

$$\tilde{C}(x) = \frac{C_1 x_2 - C_2 x_1}{x_2 - x_1} + \frac{-C_1 + C_2}{x_2 - x_1} x.$$

$$\tilde{C}(x) = \frac{x_2 - x}{x_2 - x_1} C_1 + \frac{-x_1 + x}{x_2 - x_1} C_2.$$

Construct shape functions in 1D



- Construct shape functions by writing solutions in terms of functions' values

  - For a single element

$$\tilde{C}(x) = \frac{x_2 - x}{x_2 - x_1} C_1 + \frac{-x_1 + x}{x_2 - x_1} C_2.$$

$$\tilde{C}(x) = N_1(x) C_1 + N_2(x) C_2.$$

  - Shape functions—AKA interpolation functions, basis functions for the solution:

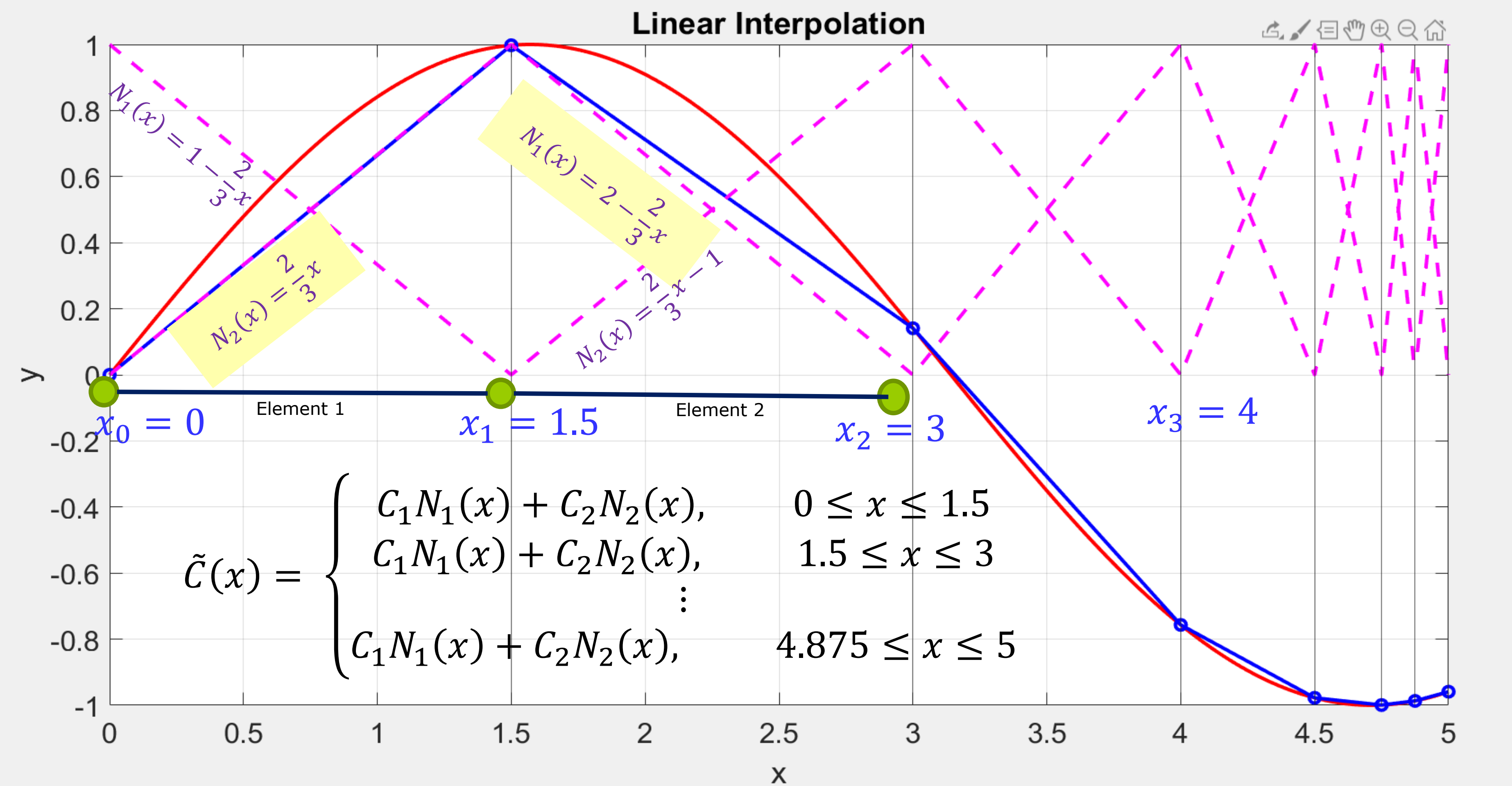
$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}, N_2(x) = \frac{-x_1 + x}{x_2 - x_1}$$

  - Solution is a linear combination of shape functions

$$\tilde{C}(x) = N_1(x) C_1 + N_2(x) C_2 = \sum_{i=1}^{2} N_i C_i = \vec{N} \cdot \vec{C}.$$

Exa 1: Interpolation

$C(x) = \sin(x)\,, 0 \le x \le 5$



Approximation includes all spatial dependence: $\tilde{C}(x) = \sum_i N_i C_i = \vec{N} \cdot \vec{C}$

- Depend on coordinates of nodes
- $N_i = 1$ at node i, $N_i = 0$ at all other nodes
- Zero outside their element

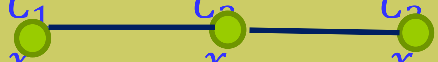
Exa 1: Interpolation--higher order shape functions

- Order 1 — Linear lagrange

2 nodes, 2 DOF

$\tilde{C}(x) = a_0 + a_1 x$

Order 2 — Quadratic line segment



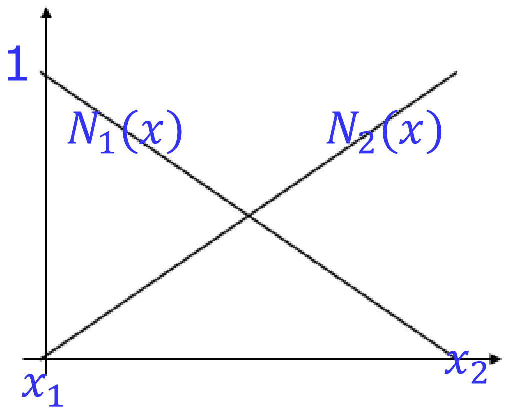
3 nodes, 3 DOF

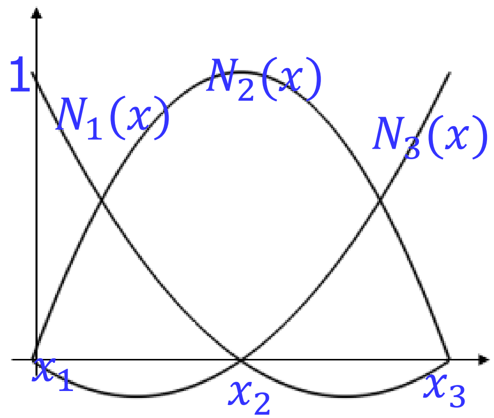
$\tilde{C}(x) = a_0 + a_1 x + a_2 x^2$

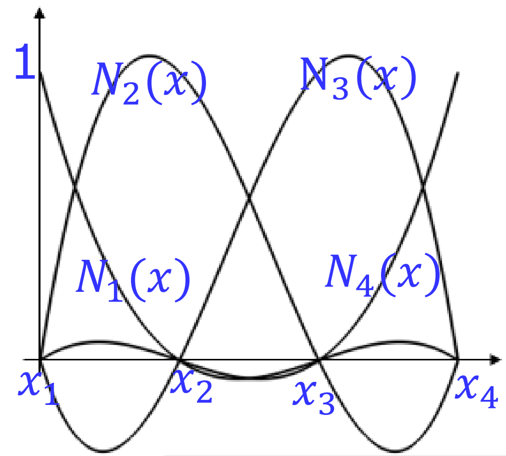
Order 3 — Cubic line segment

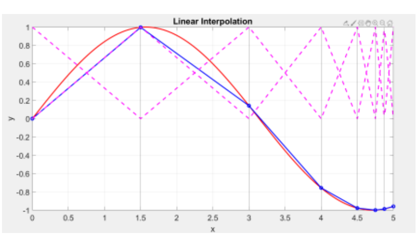
4 nodes, 4 DOF

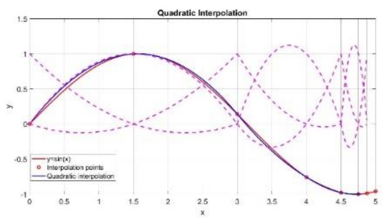
$\tilde{C}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

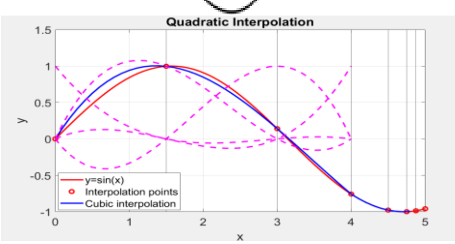






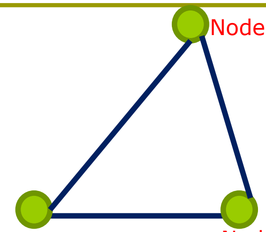






Construct shape functions in 2D

- Construct shape functions by writing solutions in terms of functions' values



Consider single element

3 nodes in 2D: $\tilde{C}(x,y) = a_0 + a_1 x + a_2 y$

Find a_0, a_1 and a_2 (3 degree of freedom DOF):

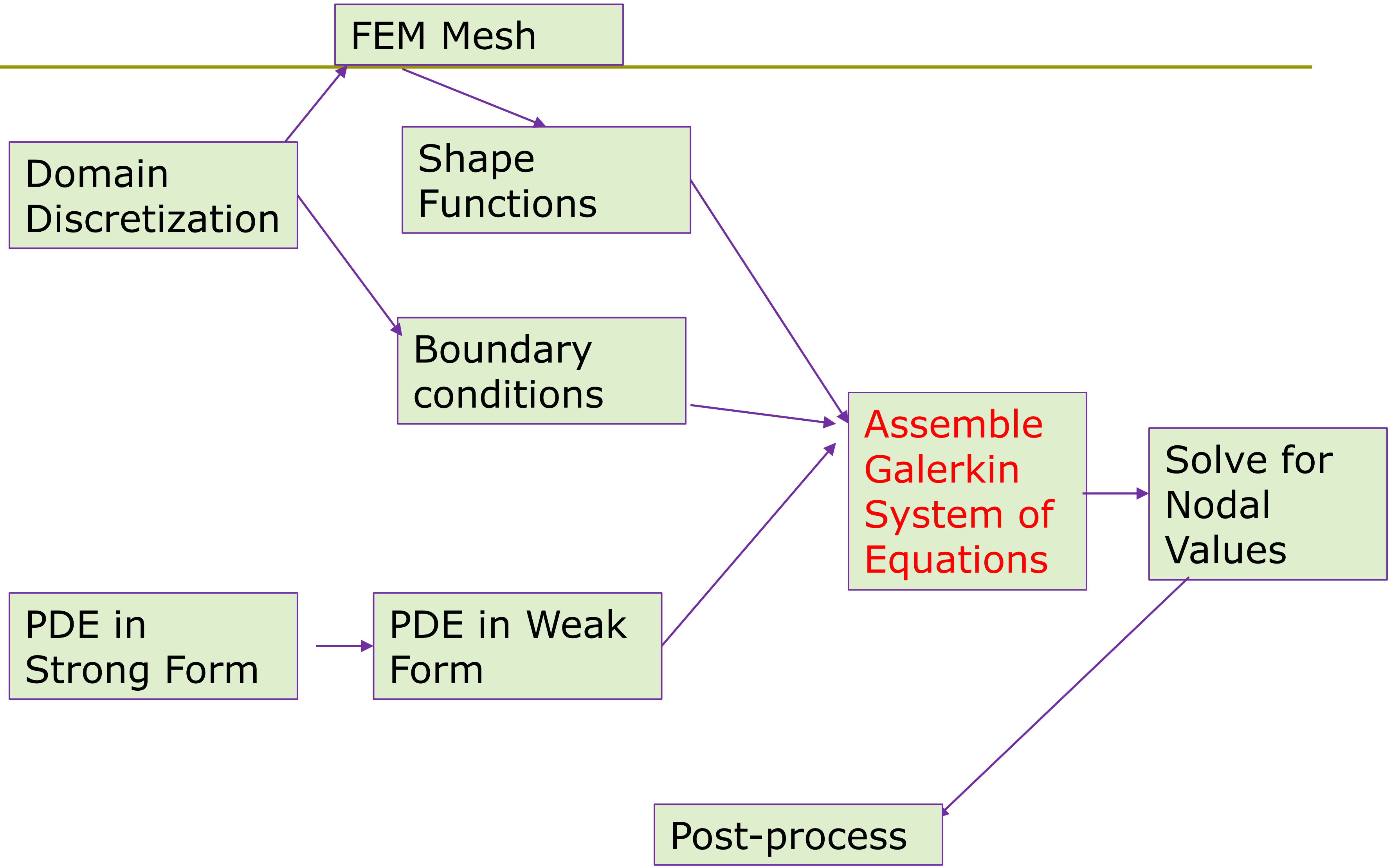
$$C_1 = \tilde{C}(x_1, y_1) = a_0 + a_1 x_1 + a_2 y_1$$
$$C_2 = \tilde{C}(x_2, y_2) = a_0 + a_1 x_2 + a_2 y_2$$
$$C_3 = \tilde{C}(x_3, y_3) = a_0 + a_1 x_3 + a_2 y_3$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \frac{1}{x_2 y_3 + x_1 y_2 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}.$$

Collect $C_1, C_2, \& \, C_3$ to get shape functions:

$$\tilde{C}(x) = N_1(x,y) C_1 + N_2(x,y) C_2 + N_3(x,y) C_3 = \sum_j N_j C_j = \vec{N} \cdot \vec{C}\,.$$

Summary of FEM Process



Exa 2: Steady state heat eqn

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

- **Strong form of PDE:**

$$\nabla \cdot \mathrm{k}\nabla \mathrm{C} - \mathrm{S} = 0 \ \text{ on } \Omega$$

Diffusion eqn

$\mathrm{k}\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - S = 0$ if k is constant

  - Step 1. Use $\tilde{\mathrm{C}} \approx \mathrm{C}$, now PDE with residual R

$$\nabla \cdot \mathrm{k}\nabla\tilde{\mathrm{C}} - \mathrm{S} = \mathrm{R} \ \text{ on } \Omega$$

  - Step 2. Want weighted error

$$\int \mathrm{w}\nabla \cdot \mathrm{k}\nabla\tilde{\mathrm{C}} - \mathrm{wS}\, d\Omega = 0.$$

Derivative twice

  - Step 3. Use Cal III identity:

$$\nabla\left(a\vec{b}\right) = \nabla a \cdot \vec{b} + a\nabla\vec{b} \text{ for } a = w, \vec{b} = \mathrm{k}\nabla\tilde{\mathrm{C}}$$

$$\int \nabla \cdot \mathrm{w\,k}\,\nabla\tilde{\mathrm{C}} - \nabla w \cdot \mathrm{k}\,\nabla\tilde{\mathrm{C}} - \ \mathrm{w\,S}\, d\Omega = 0.$$

Weaker constraints on solution

  - Step 4. Apply Divergence Theorem

$$\int \nabla \cdot \mathrm{w\,k}\,\nabla\tilde{\mathrm{C}} = \oint_{\Omega} \hat{n}_{\Omega} \cdot w\,\mathrm{k}\,\nabla\,\tilde{\mathrm{C}}\, d\Omega$$

  - Step 5. Weak form of diffusion equation

Derivative once

$$\oint_{\Omega} \hat{n}_{\Omega} \cdot w\,\mathrm{k}\,\nabla\,\tilde{\mathrm{C}}\, d\Omega - \int \nabla w \cdot \mathrm{k}\,\nabla\,\tilde{\mathrm{C}} - wS d\Omega = 0.$$

Exa 2: Boundary Condition

$D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-S=0\ if\ D$ is constant

$\nabla\cdot \mathrm{D}\nabla \mathrm{C}\ -\mathrm{S}=0$ on Ω

- Assume a tank with no leaks, i.e. zero flux BC

$$\hat{n}_{\Omega}\cdot k\,\nabla C=0.$$

- Weak form without BC

$$\oint_{\Omega}\ \hat{n}_{\Omega}\cdot w\,\mathrm{k}\,\nabla\,\tilde{\mathrm{C}}\,d\Omega-\int\nabla w\cdot \mathrm{k}\,\nabla\ \tilde{\mathrm{C}}-wSd\Omega=0.$$

- Weak form with no flux

$$\int\nabla w\cdot \mathrm{k}\,\nabla\tilde{\mathrm{C}}+wSd\Omega=0.$$

$$\mathrm{k}\int\frac{\partial w}{\partial x}\frac{\partial\tilde{\mathrm{C}}}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial\tilde{\mathrm{C}}}{\partial y}+wSd\ \Omega=0$$

- Other possible BCs:

Dirichlet, Neumann, Robin, Mixed, Cauchy

Fixed values (Dirichlet) — Flux/Natural (Neumann)

Exa 2: Galerkin Finite-Element Method

- Galerkin: $w(\vec{x}) = N_i$ for all i (all basis functions)

$$\int \nabla w \cdot \mathrm{k}\, \nabla \tilde{\mathrm{C}} + wSd\Omega = 0.$$

- Recall interpolation when given nodal values $\vec{c}$:

$$\tilde{C}(x) = \sum_j N_j C_j = \vec{N} \cdot \vec{C}$$

$$\nabla \tilde{C} = \vec{C} \cdot \nabla \vec{N} = \sum_j C_j \nabla N_j$$

$$\int \nabla N_i \cdot k \sum_j C_j \nabla N_j + N_i S d\Omega = 0$$

$$\underbrace{\int \mathrm{k} \sum_j \nabla N_i \cdot \nabla N_j \, d\Omega}_{[K]} \underbrace{C_j}_{\vec{C}} = \underbrace{-\int N_i S d\Omega}_{\vec{S}}$$

In Matrix Form

$$[K]\, \vec{C} = \vec{S}$$

- Calculate integral as sum of integral of each element:

$$\int d\Omega = \sum \int d\Omega_{element}$$

Exa 2: Steady state heat eqn -- summary

FEM Mesh

$w(\vec{x}) = N_i$ for all i

$\tilde{C}(x) = \sum_j N_j C_j = \vec{N} \cdot \vec{C}$

Domain Discretization

Shape Functions

$\nabla \tilde{C} = \vec{C} \cdot \nabla \vec{N} = \sum_j C_j \nabla N_j$

$\hat{n}_\Omega \cdot k\, \nabla C = 0.$

Boundary conditions

$\oint_\Omega \ \hat{n}_\Omega \cdot w k \nabla \tilde{C}\, d\Omega = 0$

$\int \nabla w \cdot k \nabla \tilde{C} + w S d\Omega = 0.$

$[K]\vec{C} = \vec{S}$

Assemble Galerkin System of Equations

Solve for Nodal Values

$\nabla \cdot k \nabla C - S = 0$ on Ω

$\int w \nabla \cdot k \nabla \tilde{C} - w S\, d\Omega = 0.$

PDE in Strong Form

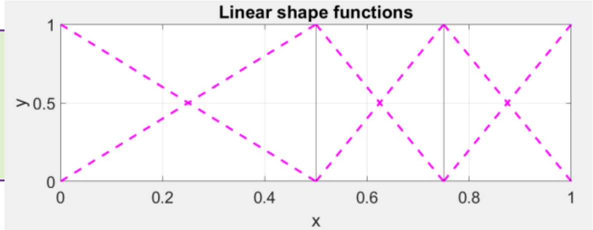
PDE in Weak Form

$\int k \sum_j \nabla N_i \cdot \nabla N_j \, d\Omega\, C_j = -\int N_i S d\Omega$

$\oint_\Omega \ \hat{n}_\Omega \cdot w\, k \nabla \tilde{C}\, d\Omega \quad -\int \nabla w \cdot k \nabla \tilde{C} - w S d\Omega = 0.$

Post-process

Exa 3: 1D Diffusion



FEM Mesh

$w(\vec{x}) = N_i$ for all i

Domain Discretization

Shape Functions

$\tilde{C}(x) = \sum_j N_j C_j = \vec{N} \cdot \vec{C}$

$\nabla\tilde{C} = \vec{C} \cdot \nabla\vec{N} = \sum_j C_j \nabla N_j$

$C(0) = C(1) = 0.$

Boundary conditions

$\int \nabla w \cdot 2\nabla\tilde{C} + wx^2 d\Omega = 0.$

$\oint_\Omega \hat{n}_\Omega \cdot w\,\tilde{C}\,d\Omega = 0$

$[K]\vec{C} = \vec{S}$

Assemble Galerkin System of Equations

Solve for Nodal Values

$\nabla \cdot 2\nabla C = x^2, 0 \le x \le 1$

$\int w\nabla \cdot 2\nabla\tilde{C} - wx^2 d\Omega = 0.$

PDE in Strong Form

PDE in Weak Form

$\int 2 \sum_j \nabla N_i \cdot \nabla N_j \, d\Omega \, C_j = -\int N_i x^2 d\Omega$

$\oint_\Omega \hat{n}_\Omega \cdot w\;\tilde{C}\,d\Omega \quad -\int \nabla w \cdot 2\nabla\,\tilde{C} - wx^2 d\Omega = 0.$

Post-process

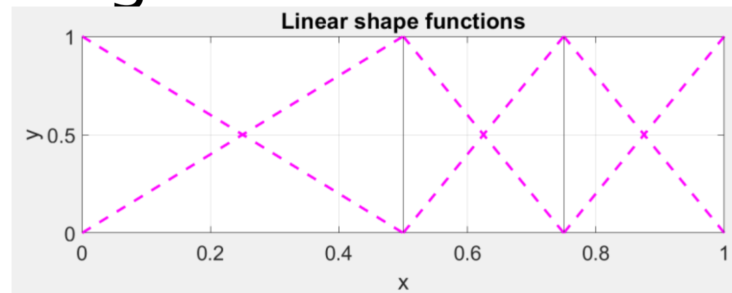
Exa 3: 1D diffusion

- Solve ODE: $\nabla \cdot 2\nabla C = x^2, 0 \le x \le 1$

$$C(0) = C(1) = 0.$$

- Galerkin linear shape function gives

$$\int 2 \sum_j \nabla N_i \cdot \nabla N_j \, d\Omega \, C_j = -\int N_i x^2 d\Omega$$



- $$\begin{cases} \sum_j \left(\int 2\, \nabla N_1 \cdot \nabla N_j \, d\Omega_1\right) C_j = -\int N_1 x^2 d\Omega_1 \\ \sum_j \left(\int 2\, \nabla N_2 \cdot \nabla N_j \, d\Omega_1\right) C_j = -\int N_2 x^2 d\Omega_1 - \int N_2 x^2 d\Omega_2 \\ \sum_j \left(\int 2\, \nabla N_3 \cdot \nabla N_j \, d\Omega_1\right) C_j = -\int N_3 x^2 d\Omega_2 - \int N_3 x^2 d\Omega_3 \\ \sum_j \left(\int 2\, \nabla N_4 \cdot \nabla N_j \, d\Omega_1\right) C_j = -\int N_4 x^2 d\Omega_3 \end{cases}$$

- $$\begin{cases} \left(\int 2\, \nabla N_1 \cdot \nabla N_1 \, d\Omega_1\right) C_1 + \left(\int 2\, \nabla N_1 \cdot \nabla N_2 \, d\Omega_2\right) C_2 = -\int N_1 x^2 d\Omega_1 \\ \left(\int 2\, \nabla N_2 \cdot \nabla N_1 \, d\Omega_1\right) C_1 + \left(\int 2\, \nabla N_2 \cdot \nabla N_2 \, d\Omega_2\right) C_2 + \left(\int 2\, \nabla N_2 \cdot \nabla N_3 \, d\Omega_3\right) C_3 = -\int N_2 x^2 d\Omega_1 - \int N_2 x^2 d\Omega_2 \\ \left(\int 2\, \nabla N_3 \cdot \nabla N_2 \, d\Omega_2\right) C_2 + \left(\int 2\, \nabla N_3 \cdot \nabla N_3 \, d\Omega_3\right) C_3 + \left(\int 2\, \nabla N_3 \cdot \nabla N_4 \, d\Omega_4\right) C_4 = -\int N_3 x^2 d\Omega_2 - \int N_3 x^2 d\Omega_3 \\ \left(\int 2\, \nabla N_4 \cdot \nabla N_3 \, d\Omega_3\right) C_3 + \left(\int 2\, \nabla N_4 \cdot \nabla N_4 \, d\Omega_4\right) C_4 = -\int N_4 x^2 d\Omega_3 \end{cases}$$

$$\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 12 & -8 & 0 \\ 0 & -8 & 16 & -8 \\ 0 & 0 & -8 & 8 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0.0104 \\ 0.0742 \\ 0.1432 \\ 0.1055 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.01823 \\ 0.01806 \\ 0 \end{bmatrix}$$

Exa 4: Heat conduction equation

$$\left(\frac{\partial \rho C T(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla T(\vec{r},t) = f\,(\vec{r},t)\right)$$

- Weak form: $\int_\Omega w \left(\frac{\partial \rho C \tilde{T}(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla \tilde{T}(\vec{r},t) = f\,(\vec{r},t)\right) d\Omega$
- Galerkin FEM: $\int_\Omega N_i(\vec{r}) \left(\frac{\partial \rho C \tilde{T}(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla \tilde{T}(\vec{r},t) = f\,(\vec{r},t)\right) d\Omega$

$$\int_\Omega N_i(\vec{r}) \left(\frac{\partial \rho C \tilde{T}(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla \tilde{T}(\vec{r},t) = f\,(\vec{r},t)\right) d\Omega$$

$$\int_\Omega N_i(\vec{r}) \frac{\partial \rho C \tilde{T}}{\partial t} d\Omega - \oint_\Omega \hat{n}_\Omega \cdot N_i(\vec{r}) k \nabla \tilde{\mathrm{T}}\, d\Omega + \int \nabla N_i(\vec{r}) \cdot \mathrm{k} \nabla \tilde{\mathrm{T}} d\Omega = \int_\Omega N_i(\vec{r}) f(\vec{r},t) d\Omega$$

$$\int_\Omega N_i(\vec{r}) \frac{\partial \rho C \tilde{T}}{\partial t} d\Omega + \int_\Omega \nabla N_i(\vec{r}) \cdot \mathrm{k} \nabla \tilde{\mathrm{T}}\, d\Omega = \int_\Omega N_i(\vec{r}) f(\vec{r},t) d\Omega + \oint_\Omega \hat{n}_\Omega \cdot N_i(\vec{r}) k \nabla \tilde{\mathrm{T}}\, d\Omega.$$

- Assume $\tilde{T}(\vec{r},t) = \sum_j N_j T_j(t)$, we have the following system of ODEs:

$$\sum_j \frac{\partial T_j(t)}{\partial t} \underbrace{\rho C \int_\Omega N_i N_j d\Omega}_{c_{ij}} + \sum_j T_j(t) \underbrace{\int_\Omega \nabla N_i \cdot \mathrm{k} \nabla N_j\, d\Omega}_{g_{ij}} = \underbrace{\int_\Omega N_i f(\vec{r},t) d\Omega + \oint_\Omega \hat{n}_\Omega \cdot N_i k \sum_j \nabla N_j\, T_j(t)\, d\Omega}_{b_i}.$$

Exa 4: Heat conduction equation

$$\left(\frac{\partial \rho C T(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla T(\vec{r},t) = f\,(\vec{r},t)\right)$$

- Weak form: $\int_\Omega w \left(\frac{\partial \rho C \tilde{T}(\vec{r},t)}{\partial t} - \nabla \cdot k\, \nabla \tilde{T}(\vec{r},t) = f\,(\vec{r},t)\right) d\Omega$
- Assume $\tilde{T}(\vec{r},t) = \sum_j N_j T_j(t)$, we have the following system of ODEs:

$$\underbrace{\sum_j \frac{dT_j(t)}{dt} \rho C \int_\Omega N_i N_j d\Omega}_{c_{ij}} + \underbrace{\sum_j T_j(t) \int_\Omega \nabla N_i \cdot \mathrm{k} \nabla N_j\, d\Omega}_{g_{ij}} = \underbrace{\int_\Omega N_i f(\vec{r},t) d\Omega + \oint_\Omega \hat{n}_\Omega \cdot N_i k \sum_j \nabla N_j\, T_j(t)\ d\Omega}_{b_i}.$$

- In matrix form: for all i,

$$\sum_j c_{ij} \frac{dT_j}{dt} + \sum_j g_{ij} T_j = b_i$$

- System of ODEs can be solved by any ODE solver, such as explicit finite difference, implicit finite difference, Backwards differentiation formula (BDF) method, Generalized alpha method, Different Runge-Kutta methods.

What to know about FEM

- Solution is a linear combination of shape functions
  - Mesh needs to match application
  - More elements improves accuracy
  - Higher order improves accuracy
- Solving system of equations takes most time
  - More DOF = more work
- Solution shouldn't depend on mesh
  - Mesh isn't real → try multiple meshes
  - User judgement → doe results make sense?
- FEM is generally used for spatial discretization

- Reference:
  - Andrew Prudil, Lecture Notes, Cybertraining Workshop at Clarkson University.

- Good read:
  - Comprehensive Introduction to Physics, PDEs, and Numerical Modeling (comsol.com)